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Freely decaying two-dimensional turbulence

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High-resolution direct numerical simulations are used to investigate freely decaying two-dimensional turbulence. We focus on the interplay between coherent vortices and vortex filaments, the second of which give rise to an inertial range. We find that Batchelor's prediction for the inertial-range enstrophy spectrum $E_\omega(k,t) \sim \beta^{2/3}k^{-1}$, where β is the enstrophy dissipation rate, is reasonably well satisfied once the turbulence is fully developed, but that the assumptions which underpin the usual interpretation of his theory are not valid. For example, the lack of a quasi-equilibrium cascade means the enstrophy flux $\Pi_\omega(k)$ is highly non-uniform throughout the inertial range, thus the common assumption that β can act as a surrogate for $\Pi_\omega(k)$ becomes questionable. We present a variant of Batchelor's theory which accounts for the wavenumber-dependence of Π_ω; in particular we propose $E_\omega(k,t) \sim \Pi_\omega(k_1)^{2/3}k^{-1}$, where k_1 is the wavenumber marking the start of the observed k^{-1} region of the enstrophy spectrum. This provides a better collapse of the data and, unlike Batchelor's original theory, can be justified on theoretical grounds. The basis for our proposal is the observation that the straining of the vortex filaments, which fuels the enstrophy flux through the inertial range, comes almost exclusively from the strain field of the coherent vortices, and this can be characterized by $\Pi_\omega(k_1)^{1/3}$. Thus $E_\omega(k)$ is a function of only k and $\Pi_\omega(k_1)$ in the inertial range, and dimensional analysis then yields $E_\omega \sim \Pi_\omega(k_1)^{2/3}k^{-1}$. We also confirm the prediction by Davidson (*Phys. Fluids*, vol. 20, 2008, 025106) that in the inertial range Π_ω varies as $\Pi_\omega(k)/\Pi_\omega(k_1) = 1 - a^{-1}\ln(k/k_1)$, where a is a constant of order 1. This corresponds to $\partial E_\omega/\partial t \sim k^{-1}$. Surprisingly, the measured enstrophy fluxes imply that the dynamics of the inertial range as defined by the behaviour of Π_ω extend to wavenumbers much smaller than k_1, but this is masked in $E_\omega(k,t)$ by the presence of coherent vortices which also contribute to E_ω in this region. In fact, we find that $kE_\omega(k,t) \approx H(k) + A(t)$, or $\partial E_\omega/\partial t \sim k^{-1}$ in this extended low-k region, where $H(k)$ is almost independent of time and represents the signature of the coherent vortices. In short, the inertial range defined by $\partial E_\omega/\partial t \sim k^{-1}$ or $\Pi_\omega(k) \sim \ln(k)$ is much broader than the observed $E_\omega \sim k^{-1}$ region.

Key words: isotropic turbulence, turbulence simulation, turbulence theory

1. Introduction

When the Reynolds number Re is sufficiently large, two-dimensional turbulence is generally observed to consist of two distinct types of structure. The large scales are usually dominated by long-lived approximately circular coherent vortices which persist and maintain their structure for many turnover times. In contrast, the small scales

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are made up of long thin filaments of vorticity which are continually stretched and distorted by the turbulent velocity field. Filaments can be produced from the large-scale vortices by various processes such as during vortex mergers, by the distortion of vortices away from a circular shape, and by the destruction of vortices in regions of strong strain. The majority of the energy in the turbulence is usually associated with the coherent vortices, whilst the fine-scale filaments are responsible for enstrophy dissipation.

It has been observed in some numerical simulations of two-dimensional turbulence at sufficiently large Re that the enstrophy spectrum $E_\omega(k)$ has a region where $E_\omega \sim k^{-1}$. This is seen in both forced (Maltrud & Vallis 1991; Lindborg & Alvelius 2000) and freely decaying (Dritschel, Tran & Scott 2007) turbulence. In order to observe a k^{-1} enstrophy spectrum a significant proportion of the domain must be covered by filaments, as coherent vortices usually lead to steeper spectra (Dritschel *et al.* 2008, and references therein). Such steep spectra are typically observed at very large times, when much of the filamentary debris has been destroyed. In this paper, we restrict ourselves to earlier times where the k^{-1} enstrophy spectrum appears to be a robust phenomenon. This spectrum is widely believed to represent a direct enstrophy cascade as proposed by Batchelor (1969) for freely decaying turbulence and Kraichnan (1967) for forced turbulence. However, we shall see that, particularly for freely decaying turbulence, the usual arguments leading to the k^{-1} spectrum are flawed and cannot satisfactorily explain its existence.

Batchelor's theory is a two-dimensional analogue of Kolmogorov's 5/3 law, which predicts the shape of the energy spectrum $E(k)$ in three-dimensional turbulence (Kolmogorov 1941). Batchelor assumed that in the limit of $Re \to \infty$ the energy dissipation rate, ϵ, tends to zero, but the enstrophy dissipation rate β remains finite due to the generation of large vorticity gradients by the creation of thin filaments, the smallest of which takes a time of order $\langle \omega^2 \rangle^{1/2} t \sim \log Re$ to form (note that Tran & Dritschel 2006 have shown that if t is fixed while $Re \to \infty$, then $\beta \to 0$). Since the enstrophy dissipation takes place primarily at the small scales, he proposed that enstrophy cascades from large to small scales. In common with Kolmogorov's hypothesis, he assumed that at scales sufficiently removed from the integral scale ℓ, the only relevant parameters are the viscosity and the enstrophy dissipation rate β. Dimensional analysis then gives the form of the enstrophy spectrum

$$E_\omega = \beta^{1/2} \nu^{1/2} F(k \nu^{1/2} / \beta^{1/6}), \qquad (1.1)$$

where F is assumed to be a universal function. If there exists a range of scales which are sufficiently removed from the integral scale for (1.1) to apply but large enough to be unaffected by viscosity (an inertial range), then ν cannot be a relevant parameter and (1.1) reduces to

$$E_\omega = C \beta^{2/3} k^{-1}, \qquad (1.2)$$

where C is a universal constant.

The assertion that the inertial range is controlled by only one external parameter, and that this parameter is β, is a bold hypothesis. It is natural to ask how this may be justified. Although not explicitly discussed in Batchelor (1969), there are a number of implicit assumptions inherent in (1.1) and (1.2). A hint that this is so comes from the fact that Batchelor repeatedly refers to this theory as a universal equilibrium theory of the small scales, analogous to Kolmogorov's celebrated universal equilibrium theory of three-dimensional turbulence. To understand what Batchelor meant by the terms 'universal' and 'equilibrium' we must go back to Batchelor (1953, chapter 6), where he makes it quite explicit what these two terms mean and what their significance is. Let us start with the term 'universal'. Here Batchelor argues that, in three dimensions, a cascade is an information-losing process, so that scales of very different wavenumber should be statistically independent (see Batchelor 1953, p. 110). This is important because it suggests that the small scales are independent of the large eddies, and in particular they will not depend explicitly on the integral scales u and ℓ. Note that Batchelor makes the same hypothesis for two-dimensional turbulence in his 1969 paper, asserting statistical decoupling of remote wavenumbers through a cascade process (Batchelor 1969, § I). The term 'equilibrium' is also discussed in detail in Batchelor (1953), where it is noted that in three-dimensional turbulence the small scales evolve rapidly relative to the large scales, so that at any instant the small scales can be considered to be in statistical equilibrium relative to the large scales (Batchelor 1953, p. 104). Batchelor then argues that the dual concepts of small-scale universality and small-scale equilibrium are the central hypotheses which underpin Kolmogorov's theory, since taken together they imply that the only external parameters which can determine the statistics of the equilibrium range are the energy flux entering that range and the energy flux leaving that range (Batchelor 1953, p. 114). The final step in Batchelor's argument is to note that, because the small scales in three-dimensional turbulence are in statistical equilibrium, the energy flux entering the inertial-range cascade is equal to that leaving, which in turn equals the energy dissipation rate, ϵ (Batchelor 1953, p. 122). In short, Kolmogorov's hypothesis that, in three dimensions, ϵ is the only external parameter controlling the inertial-range statistics was, in Batchelor's opinion, crucially dependent on the dual concepts of universality of the small scales (because greatly different scales are statistically independent) and statistical equilibrium of the small scales (which means the dissipation ϵ can be used as a surrogate for the inertial-range flux).

In his seminal 1969 paper on two-dimensional turbulence many of these details are skipped over, but it seems likely that he had in mind a similar picture for the enstrophy cascade. That is to say: (i) the inertial-range statistics should not depend explicitly on the integral scales u and ℓ, because remote wavenumbers should be statistically independent; and (ii) statistical equilibrium of the inertial range means that the only external parameters controlling that range are the enstrophy flux entering and leaving the inertial range, for which β can act as a surrogate. Strong hints that this was indeed the underlying logic come from the fact that Batchelor explicitly asserts statistical decoupling of remote wavenumbers, repeatedly refers to his theory as an equilibrium theory and at one point in § II uses β as a surrogate for enstrophy flux.

In any event, whatever the motivation of Batchelor (1969), the assertion that β, and only β, controls the inertial-range statistics needs to be justified. For example, how can we explain that the inertial-range statistics are independent of the integral scales u and ℓ, without some kind of statistical decoupling of small and large scales? Moreover, how can the inertial-range statistics, which are assumed to be independent of all viscous dynamics, be an explicit function of the dissipation-scale process β, unless β is being used as a surrogate for the inviscid enstrophy flux, $\Pi_\omega(k)$? While these questions seem to be untroubling when posed in their equivalent three-dimensional context (for the reasons given above), they present profound difficulties in two-dimensional turbulence, as we now discuss.

The first point to note is that there is an internal inconsistency in Batchelor's assumption that remote wavenumbers are statistically independent. The mean square strain or shear at wavenumber k can be approximated as $\int_0^k E_\omega(k')\,dk'$ (Kraichnan 1971), so for $E_\omega \sim k^{-1}$ each decade of the spectrum contributes equally to the strain at a given scale, implying that structures of very different sizes can interact. In this respect, two-dimensional turbulence is markedly different from three-dimensional turbulence and the reason can be traced back to the absence of vortex stretching in two dimensions. In three-dimensional turbulence the smaller eddies have more intense vorticity and hence stronger strain, leading to interactions which are local in scale-space. This possibility is excluded in two dimensions by the material conservation of vorticity, and interactions between structures of very different sizes are likely to be important. The importance of non-local interactions between large-scale coherent vortices and fine-scale filaments has been suggested by several authors, for example, Brachet *et al.* (1988) and Oetzel & Vallis (1997). Loosely speaking, the strain acting on the filaments comes directly from the large-scale vortices, and there is very little in the way of filaments straining other filaments. In short, the filaments are quasi-passive, and interactions between structures of very different scales are important.

The material conservation of vorticity in two dimensions also removes the justification for assuming that the cascade is in a state of quasi-equilibrium, and hence that β can be used as a surrogate for the enstrophy flux. The time scale for eddies of scale L is ω_L^{-1}, where ω_L is the characteristic vorticity of eddies of scale L; since eddies of all scales larger than the dissipation scales will have the same characteristic vorticity, they therefore evolve on the same time scale. In contrast, in three dimensions the more intense vorticity of the smaller vortices leads to faster evolution times and hence the possibility of a quasi-equilibrium cascade. The lack of a quasi-equilibrium cascade removes the justification for assuming a near-uniform inertial-range flux, and if Π_ω is a function of k, then β can no longer be used as a surrogate for both the enstrophy flux entering the cascade and that leaving it. Indeed, Davidson (2008) has considered a simple model of freely decaying two-dimensional turbulence where the inertial-range enstrophy spectrum is assumed to take the form

$$E_\omega = A(t)k^{-1}, \quad k_1 < k < k_2, \tag{1.3}$$

and demonstrated that the resulting enstrophy flux is wavenumber-dependent, with $\Pi_\omega(k_1) \neq \beta$. We shall summarize his model and predictions here, as they illustrate important features of freely decaying two-dimensional turbulence which will prove to be useful later.

Substituting (1.3) into the spectral form of the two-dimensional Kármán–Howarth equation,

$$\frac{\partial}{\partial t}(E_\omega(k,t)) = -\frac{\partial}{\partial k}(\Pi_\omega(k,t)), \tag{1.4}$$

yields the enstrophy flux

$$\Pi_\omega(k) = \Pi_\omega(k_1) - A'(t)\ln\left(\frac{k}{k_1}\right). \tag{1.5}$$

The wavenumber-dependence of the enstrophy flux is evidently determined by the magnitude of $A'(t)$, and Davidson's model suggests that this is non-negligible. By assuming that the enstrophy spectrum falls off rapidly for $k > k_2$, a simple balance of energy and enstrophy dissipation (accounting for any time-dependence of k_2) implies that $\Pi_\omega(k_2) = k_2^2 \Pi_u(k_2)$, where Π_u is the the energy flux. Davidson also assumed that at wavenumber k_1 the relationship between the enstrophy and energy flux can be expressed as

$$\Pi_\omega(k_1) = 2ak_1^2 \Pi_u(k_1), \tag{1.6}$$

where a is an order 1 constant which can be either positive or negative. It can be shown that, for $k_2 \gg k_1$, these relationships imply $\Pi_\omega(k_1) = aA'(t) + O(k_1^2/k_2^2)$, giving an enstrophy flux for $k_1 < k < k_2$:

$$\frac{\Pi_\omega(k)}{\Pi_\omega(k_1)} = 1 - \frac{1}{a} \ln\left(\frac{k}{k_1}\right). \tag{1.7}$$

We therefore expect that for an extended k^{-1} inertial range the enstrophy flux will have a logarithmic dependence on k, and if Re is sufficiently large the variation in flux will be non-negligible. Furthermore, the magnitude of the logarithmic correction is determined by the relationship between the fluxes of energy and enstrophy at wavenumber k_1. Note that, in a simulation, we will take k_1 to be defined by the start of the observed $E_\omega \sim k^{-1}$ region. In fact, as discussed in § 4, it does not matter what value is taken for k_1, as long as it lies within the inertial range.

The results of Davidson (2008) suggest that the enstrophy flux is not uniform throughout the inertial range, and this will have important implications for Batchelor's theory. For example, if we follow Batchelor (1953) and assume that the only important external parameters in the inertial range are the flux in and out of that range, and only one of these is equal to β, how can we justify a theory in which β is the only relevant parameter? We could, for example, equally adopt $\Pi_\omega(k_1)$ as the control parameter.

It appears, therefore, that the assumptions which underpin the usual interpretation of Batchelor's theory are questionable and warrant further investigation. In this paper we use high-resolution direct numerical simulations to study the enstrophy cascade in freely decaying two-dimensional turbulence. We show that at sufficiently high Re an $E_\omega \sim k^{-1}$ region does indeed exist, but that the assumptions of local interactions and a uniform enstrophy flux are not well satisfied. In fact, the large-scale vortices are essential to the dynamics of the k^{-1} spectrum, acting directly on the small-scale filaments. Moreover, the enstrophy flux is strongly wavenumber-dependent, in accordance with (1.7). Nevertheless, we find that Batchelor's scaling (1.1) still provides an excellent description of the evolution of the enstrophy spectrum at high k, and we have explained this apparent contradiction. The structure of the paper is as follows: in § 2 we outline the details of our simulations and the numerical methods used, and in § 3 we present our results. Our interpretation of the numerical findings is given in § 4, and our conclusions are summarized in § 5.

2. Overview of the simulations

We used standard pseudospectral methods to numerically integrate the equations

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\boldsymbol{u}\omega) = (-1)^{p+1} \nu_p \nabla^{2p} \omega, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{2.1}$$

where ω is the vorticity, which is related to the velocity by $\omega \hat{\boldsymbol{e}}_z = \nabla \times \boldsymbol{u}$. When $p = 1$ the diffusion term represents the usual Newtonian viscosity, whilst for $p > 1$ it represents hyperviscosity. The initial conditions consisted of Fourier modes with random phases and a prescribed energy spectrum $E(k) \sim k^3 \exp(-3k^2/2k_p^2)$, which has a maximum at $k = k_p$, and the velocity was scaled to make the initial energy $1/2\langle \boldsymbol{u}^2 \rangle = 1/2$. We present results from two separate runs. The first, run N, used standard Newtonian viscosity ($p = 1$), had a resolution of 8192^2 grid points and an initial energy spectrum

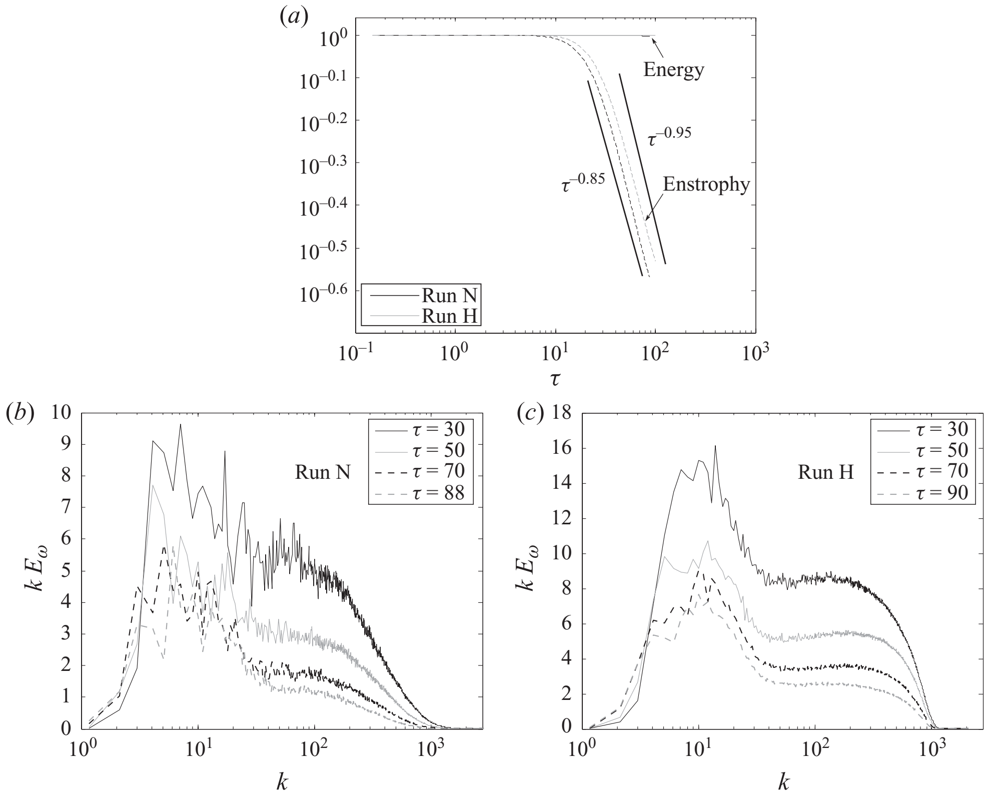

FIGURE 1. The evolution of (*a*) energy and enstrophy normalized by their initial values, and (*b*, *c*) the enstrophy spectra in runs N and H.

with $k_p = 8$. The second, run H, used hyperviscosity with $p = 4$, 4096^2 grid points and $k_p = 10$. We performed 10 realizations of run H with different initial conditions in order to obtain ensemble averages.

For run N we used the algorithm outlined in Fox & Davidson (2008). In run H, time stepping was performed using a leap-frog scheme with a weak Robert filter for the advection term, and a Crank–Nicholson scheme for the diffusive term. Dealiasing in run H was performed using the usual 2/3 rule.

We define an inverse time scale ω_0 based on the initial enstrophy as $\omega_0 = \langle \omega^2 \rangle^{1/2}$ and a non-dimensional time $\tau = \omega_0 t$. Run N was integrated up to $\tau = 88$, whilst run H was integrated until $\tau = 100$. In both runs ν_p was chosen to give a well-resolved dissipation range, and the initial Reynolds number for run N, defined as $Re = \omega_0/(k_p^2 \nu)$, is 2.9×10^4.

3. Results

Figure 1 shows the evolution of the energy, enstrophy and enstrophy spectra (in compensated form) in runs N and H. It can be seen that $\langle \omega^2 \rangle$ falls off as $\langle \omega^2 \rangle \sim \tau^{-n}$ where $0.85 < n < 0.95$, which is consistent with other studies (see Lowe & Davidson 2005, and references therein). Also, it is clear that after an initial transient of $\tau \sim 30$ a definite k^{-1} region forms. Note that E_ω is pre-multiplied by k so that the k^{-1} region appears as a plateau. This allows the use of a linear scale for the vertical axis and

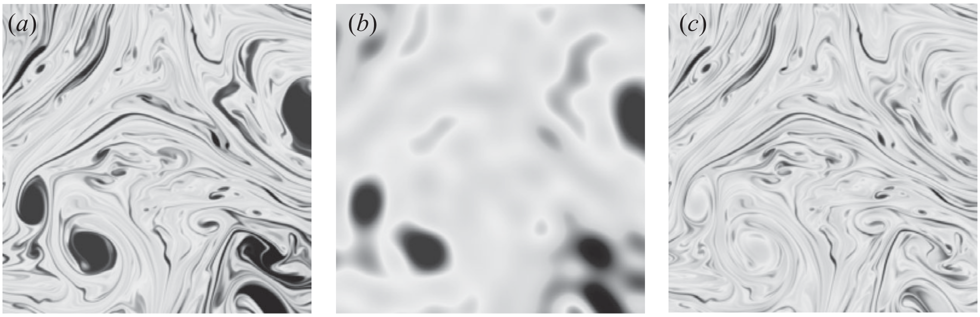

FIGURE 2. A subset of the vorticity field at $\tau = 50$ in run N. The area shown is 1/16 of the total area. (*a*) Full vorticity field, (*b*) low-pass filtered vorticity and (*c*) high-pass filtered vorticity. The filter cutoff is $k = 35$.

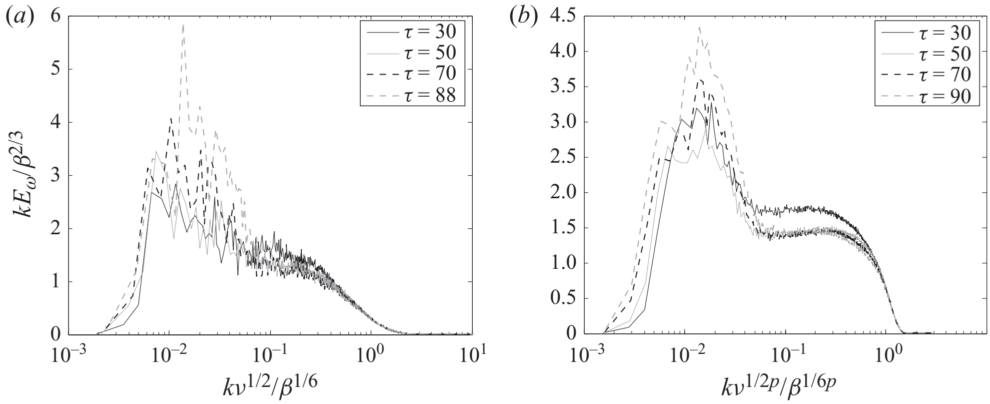

FIGURE 3. The evolution of the enstrophy spectrum scaled according to Batchelor's theory in (*a*) run N and (*b*) run H.

provides a stringent test for an $E_\omega \sim k^{-1}$ power law. Note also the very large values of *Re* required for this region to occupy a significant range of wavenumbers, which perhaps accounts for the steeper spectra observed in many previous simulations. To the right of the k^{-1} region the spectrum falls off rapidly corresponding to the dissipation range, whilst to the left there is a pronounced hump. The hump is believed to be due to the coherent vortices, whilst the k^{-1} range corresponds to the filaments. Some indication of this can be seen by both high- and low-pass filtering the vorticity field with a cutoff wavenumber corresponding to the start of the k^{-1} region. This is shown using a sharp spectral filter at $k = 35$ for run N at $\tau = 50$ in figure 2. It can be seen that the approximate k^{-1} region is indeed dominated by the filaments, whilst the coherent vortices are primarily responsible for the enstrophy spectrum at lower wavenumbers.

Having demonstrated the existence of a k^{-1} enstrophy spectrum in our simulations, we now test Batchelor's scaling (1.1) and its hyperviscous equivalent. Figure 3 shows the enstrophy spectra for the times which display a k^{-1} region scaled accordingly, and it can be seen that the scaling works remarkably well and gives a good collapse of both the inertial and dissipation ranges (the curves in figure 3*b* are smoother than those of 3*a* because these are ensemble-averaged runs). There is a single anomalous result corresponding to $\tau = 30$ in both runs; as we shall see, this is due to the fact that

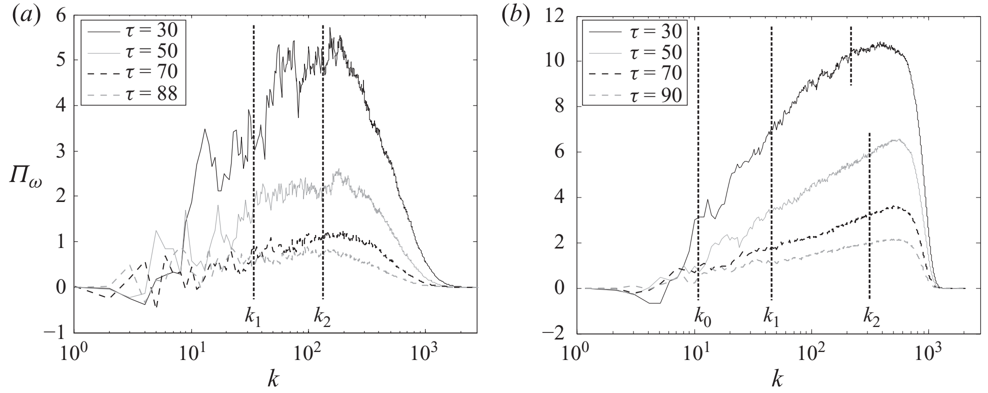

FIGURE 4. The enstrophy flux, Π_ω in (*a*) run N and (*b*) run H. k_1 and k_2 mark the limits of the k^{-1} region in the corresponding enstrophy spectrum E_ω.

the small scales of the turbulence are not fully developed at this time, and hence the enstrophy dissipation rate does not give a good estimate of the flux. The numerical value of C is also comparable between the two cases: 1.3 for run N and 1.4 for run H.

At first sight, it appears that Batchelor's theory is sufficient to explain the emergence of a k^{-1} enstrophy spectrum in freely decaying two-dimensional turbulence. However, it rests on the hypothesis that β can be used as a surrogate for the enstrophy flux Π_ω, which in turn requires that Π_ω is uniform throughout the inertial range, and we have seen that the usual justification for these assumptions is unlikely to hold in practice. Figure 4 shows Π_ω in our simulations for the times where the enstrophy spectrum shows a significant k^{-1} region. The vertical lines $k = k_1$ and $k = k_2$ represent the limits of the $E_\omega \sim k^{-1}$ range, and the significance of the line $k = k_0$ will be explained shortly. It is clear from the run H results that the enstrophy flux is not constant throughout the inertial range, and increases with increasing wavenumber. Strikingly, the variation in Π_ω throughout the k^{-1} region is not small, and is consistent with the $\log k$ dependence predicted by (1.5). The trend is less obvious in run N, presumably due to the shorter inertial range and the lack of ensemble averaging. Nevertheless, there is still some indication of the enstrophy flux increasing at higher wavenumbers.

A wavenumber-dependent enstrophy flux does not necessarily invalidate the central assumption of Batchelor's hypothesis that β is the only relevant dynamical parameter in the inertial range. It does, however, make it seem most improbable. One obvious modification to Batchelor's theory is to replace β by $\Pi_\omega(k)$, giving

$$E_\omega(k) = C\Pi_\omega(k)^{2/3}k^{-1}. \tag{3.1}$$

However, since Π_ω is a function of k (3.1) no longer predicts a k^{-1} enstrophy spectrum. Figure 5 shows both the compensated spectrum kE_ω and the form predicted by (3.1) using the measured flux, i.e. $C\Pi_\omega(k)^{2/3}$. We have used a value of $C = 2$ as this makes the magnitudes similar in the inertial range. It is clear that the results are not consistent, so we must discard (3.1) as a modified theory and search for some alternative.

It appears that the success of Batchelor's scaling demonstrated in figure 3 is somewhat coincidental, as we have shown that the underlying assumptions leading to (1.1) are not well satisfied. We do, however, observe a k^{-1} enstrophy spectrum, so our results should be consistent with the predictions of the model of Davidson (2008) which were outlined above; we now confirm this.

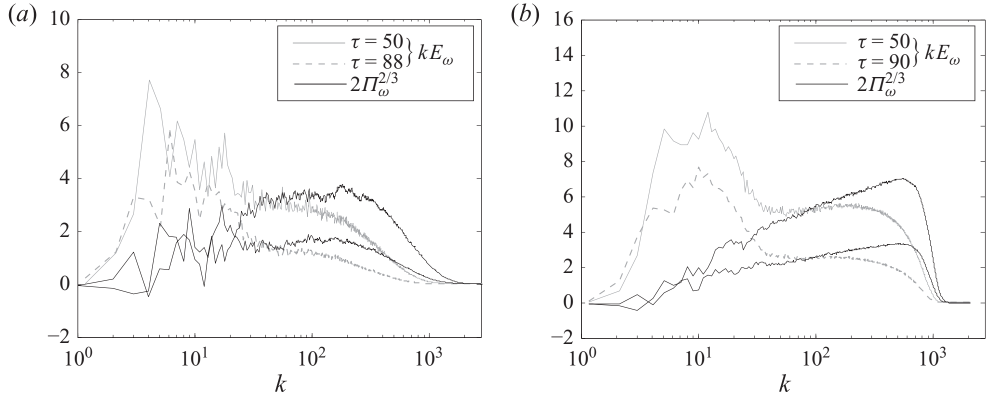

FIGURE 5. The compensated enstrophy spectrum kE_ω, and the compensated spectrum implied by (3.1), $C\Pi_\omega^{2/3}$ in (*a*) run N and (*b*) run H for $C = 2$.

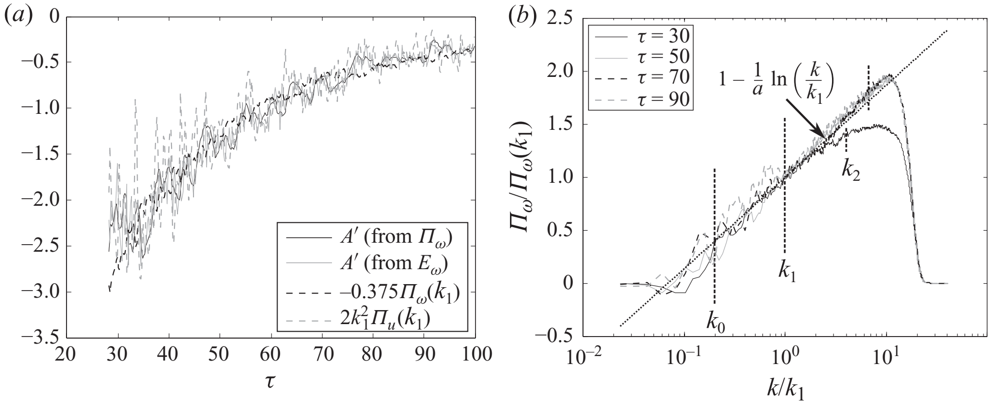

FIGURE 6. (*a*) The variation of $A'(t)$, $\Pi_\omega(k_1)$ and $\Pi_u(k_1)$ with time in run H. (*b*) The enstrophy flux scaled according to (1.7).

We first demonstrate that the measured fluxes are consistent with (1.5), which predicts a logarithmic variation of the flux in the inertial range. This should appear as a straight line in the semilogarithmic plots of figure 4, and that is indeed what is observed for run H. The slope of the line should give $A'(t)$, and this is confirmed in figure 6(*a*), where the slope is compared with a finite difference approximation to $A'(t)$, with $A(t)$ measured directly from the k^{-1} region of the enstrophy spectrum.

Davidson's model also predicts that $A'(t) = a^{-1}\Pi_\omega(k_1, t)$, where a is defined by (1.6) and is a non-dimensional measure of the ratio of energy flux to enstrophy flux at wavenumber k_1. For a fixed value of $k_1 = 50$, which is approximately the location of the start of the k^{-1} region in figure 1(*b*), we find that $A'(t) = -0.375\Pi_\omega(k_1, t)$ (figure 6*a*). This implies that a is constant throughout run H, and has a numerical value of approximately -2.67. We also demonstrate in figure 6(*a*) that this numerical value of a is consistent with definition (1.6), by comparing $a^{-1}\Pi_\omega(k_1)$ with $2k_1^2\Pi_u(k_1)$. Equation (1.7) suggests that the non-dimensional enstrophy flux $\Pi_\omega(k)/\Pi_\omega(k_1)$ should be a universal function of k/k_1 for $k_1 < k < k_2$. We plot this in figure 6(*b*), and it can be seen that this is indeed the case.

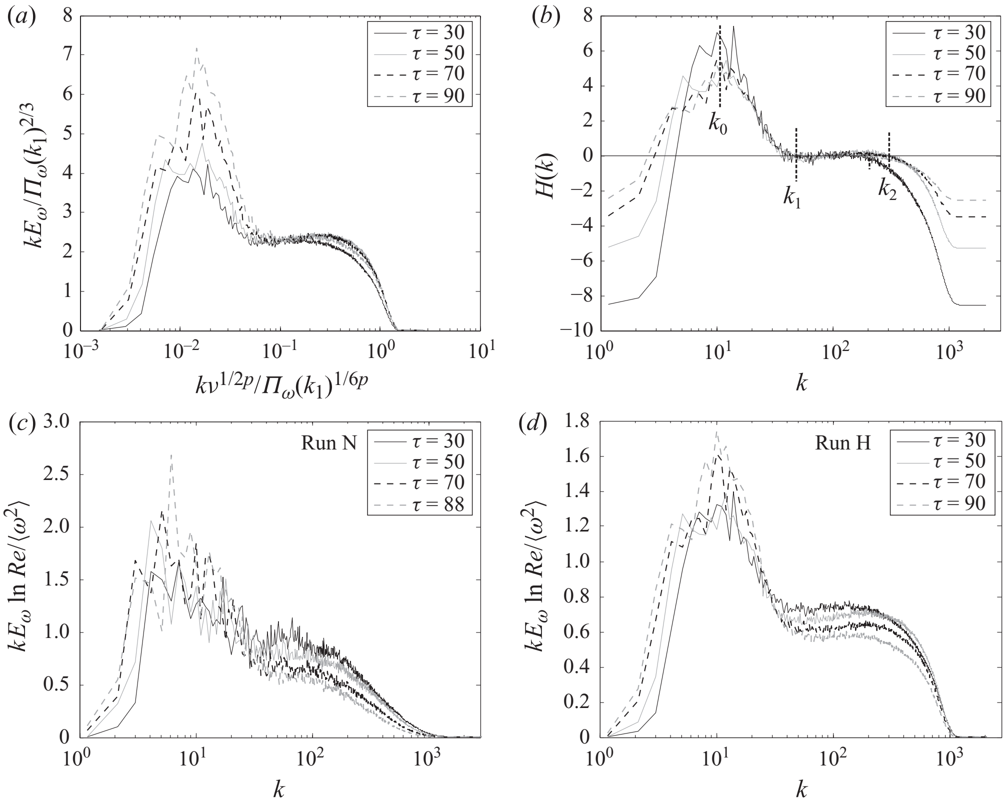

FIGURE 7. From run H: (a) the compensated enstrophy spectrum scaled by the enstrophy flux at k_1, (b) $H(k) = kE_\omega - A(t)$ and (c, d) the scaling for E_ω proposed by Dritschel *et al.* (2007).

It is also clear from figure 6(b) that at $\tau = 30$ the inertial range is not fully developed, since the flux flattens off at a much smaller wavenumber. This explains why the enstrophy spectrum at this time did not collapse well under Batchelor's scaling in figure 3. We now try an experiment and replace β by $\Pi_\omega(k_1)$ in Batchelor's scaling, i.e.

$$E_\omega \sim \Pi_\omega(k_1)^{2/3} k^{-1}. \tag{3.2}$$

It can be seen from figure 7(a) that this gives an excellent collapse of the inertial-range spectrum at all times when there is a k^{-1} region, including $\tau = 30$. Evidently the key parameter in Batchelor's scaling is $\Pi_\omega(k_1)$ and not β. Note that (3.2) is different from the scaling proposed by Dritschel *et al.* (2007), i.e.

$$E_\omega \sim \frac{\langle\omega^2\rangle}{\ln Re} k^{-1}, \tag{3.3}$$

and indeed (3.2) gives a better collapse of the data (see figure 7c,d, where we have taken $Re = k_\nu^2/k_0^2$, with $k_0 = \langle\omega^2\rangle^{1/2}/\langle \boldsymbol{u}^2\rangle^{1/2}$ and $k_\nu = \langle\omega^2\rangle^{1/4p}/\nu^{1/2p}$).

There is another rather more surprising feature of the enstrophy fluxes plotted in figure 6(b): it appears that the logarithmic region extends to wavenumbers $k < k_1$, say $k_0 < k < k_2$, and in fact covers approximately two decades in wavenumber space as opposed to the single decade occupied by the k^{-1} enstrophy spectrum. This extended logarithmic region requires the enstrophy spectrum to have the form

$$kE_\omega(k,t) = A(t) + H(k), \quad k_0 < k < k_2, \tag{3.4}$$

where $H(k)$ is independent of time, as can be seen from (1.4).

We can gain further verification that this is a reasonable description of our results by subtracting $A(t)$ from the compensated enstrophy spectrum $kE_\omega(k)$ to give $H(k)$. This is plotted in figure 7(b), and it can be seen that there is indeed a range of wavenumbers $k_0 < k < k_2$ where $H(k)$ is approximately independent of time. We have reached a surprising conclusion: if we define the inertial range to be that region in which $\partial E_\omega/\partial t \sim k^{-1}$, or equivalently where Π_ω is given by (1.5), then it extends over a much wider range than the $E_\omega \sim k^{-1}$ region. Now we have already seen in figure 2 that the coherent vortices occupy the wavenumber range $k < k_1$, so it seems likely that the presence of the coherent vortices masks the underlying dynamics of the inertial range in the region $k_0 < k < k_1$. In short, $H(k)$ is the signature of the coherent vortices in this region. Note that, since we would expect some changes to the population of coherent vortices over very long time scales, $H(k)$ may be weakly time-dependent, but this is not evident in our simulations.

4. Dynamics of the enstrophy cascade

We saw in the previous section that our numerical simulations demonstrate a clear k^{-1} enstrophy spectrum over an extended range of wavenumbers. We also found that the enstrophy flux is wavenumber-dependent, but that it has a universal form once the k^{-1} enstrophy spectrum has developed. A wavenumber-dependent $\Pi_\omega(k)$ raises questions as to whether a single surrogate, such as β, can be used to represent the inertial-range flux in any theory. However, figure 7(a) shows that in spite of this, Batchelor's scaling with the enstrophy dissipation β replaced by $\Pi_\omega(k_1)$ provides an excellent description of our results. In this section we present an explanation of these results.

Our starting point is to assume that the dynamics of freely decaying two-dimensional turbulence are driven by the coherent vortices which exist at the large scales. In making this assumption we appeal to the phenomenological picture of fine-scale filaments being passively teased out by the strain field caused by the large coherent vortices. Some indication that this is a reasonable assumption can be found by calculating the enstrophy flux caused by the large-scale velocity field. We define a low-pass filtered velocity field $\widetilde{\boldsymbol{u}}$, and use it to calculate the modified enstrophy flux $\widetilde{\Pi}_\omega(k) = \sum_{k'<k} \mathrm{Re}(\widehat{\omega}^\dagger \widehat{\nabla\cdot(\widetilde{\boldsymbol{u}}\omega)})$, where Re represents the real part, $(\sim)^\dagger$ represents the complex conjugate and $\widehat{(\sim)}$ represents the Fourier transform. We have used a sharp spectral filter with a cutoff at $k = 50$ to calculate $\widetilde{\boldsymbol{u}}$, and compare the modified flux to the true flux at $\tau = 50$ for run H in figure 8. It can be seen that they are in good agreement over a wide range of wavenumbers, although there is some difference at very large values of k. We therefore conclude that it is reasonable to assume that it is mainly the strain field generated by the large coherent vortices which drives the enstrophy cascade.

Continuing with our phenomenological cartoon of quasi-passive fine-scale filaments being stretched and distorted by the strain of the coherent vortices, we consider what features of the large scales will be relevant for the small-scale enstrophy cascade. There appears to be two important processes: the generation of filaments, which results in an injection of enstrophy into the cascade, and the continual thinning of existing filaments which leads to this enstrophy being carried down to smaller scales.

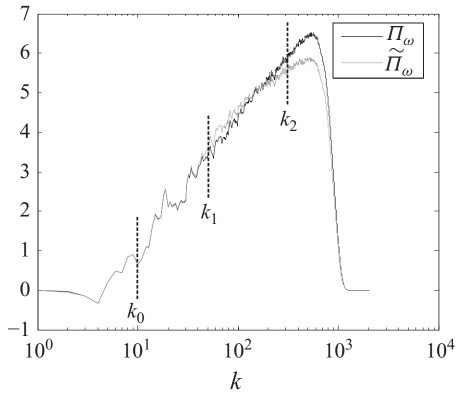

FIGURE 8. Comparison between the true enstrophy flux and the modified enstrophy flux with a filter scale $k_c = 50$ at $\tau = 50$ in run H.

However, it is not inconceivable that a single parameter can be used to characterize both these processes. The thinning of filaments is due to the strain from the coherent vortices, and it is this same strain that results in new filaments being formed, usually when weaker vortices are caught up in the strain surrounding strong vortices.

Since we anticipate that a single large-scale parameter will control the rate of injection of enstrophy into the cascade, we choose to use $\Pi_\omega(k_1)$ as our large-scale parameter, where k_1 is the lower end of the observed $E_\omega \sim k^{-1}$ range. Of course, the choice of k_1 is a little arbitrary, as we have seen that the true $kE_\omega(k,t) \sim A(t) + H(k)$ range extends to lower values of k, in fact down to k_0. However, (1.7) tells us that it does not matter whether we choose $\Pi_\omega(k_1)$ or $\Pi_\omega(k_1')$, with $k_0 < k_1' < k_1$, as the two are related by $\Pi_\omega(k_1') = [1 - (1/a)\ln(k_1'/k_1)]\Pi_\omega(k_1)$, and k_1'/k_1 is approximately constant during the decay. That is to say, $\Pi_\omega(k_1')$ is proportional to $\Pi_\omega(k_1)$, so it does not matter which we choose as our large-scale parameter. (We note in passing that $\Pi_\omega(k)$ must go negative at low k, as pointed out by Davidson 2008. If k_1 is chosen to lie in this region, then the sign of a changes from negative to positive.)

With the assumptions that the enstrophy cascade is controlled by the large scales, and that their effect can be reduced to a single parameter $\Pi_\omega(k_1)$, or $\Pi_\omega(k_1')$, we arrive at the hypothesis that $E_\omega = G(k, \Pi_\omega(k_1))$, and dimensional analysis then gives $E_\omega = C'\Pi_\omega(k_1)^{2/3}k^{-1}$, which we found to be in excellent agreement with our results (figure 7*a*).

Note that we can re-write our proposed spectrum as $E_\omega \sim \Pi_\omega(k_1)S_{cv}^{-1}k^{-1}$, where $S_{cv} \sim [\Pi_\omega(k_1)]^{1/3}$ is the characteristic strain rate associated with the coherent vortices. There is strong similarity between this line of reasoning and that which leads to the high-Prandtl-number passive-scalar spectrum $E_{ps} \sim \Pi_{ps}S_K^{-1}k^{-1}$ at scales smaller than the Kolmogorov scale but larger than the scalar dissipation scale. Here, E_{ps} is the passive-scalar spectrum, Π_{ps} is the flux of the passive scalar to small scales, and S_K is the r.m.s. strain field established by the Kolmogorov-scale eddies, which is the analogue of S_{cv} in two-dimensional turbulence. In this analogy, the Kolmogorov eddies play the role of the coherent vortices, and the passive scalar is the analogue of our passive vortex filaments.

This cartoon consisting of the quasi-passive straining of vortical structures also suggests the interpretation of the extended inertial range for wavenumbers $k_0 < k < k_1$. Recall that this range of wavenumbers is occupied by the coherent vortices which are responsible for driving the enstrophy cascade. However, if these large-scale structures

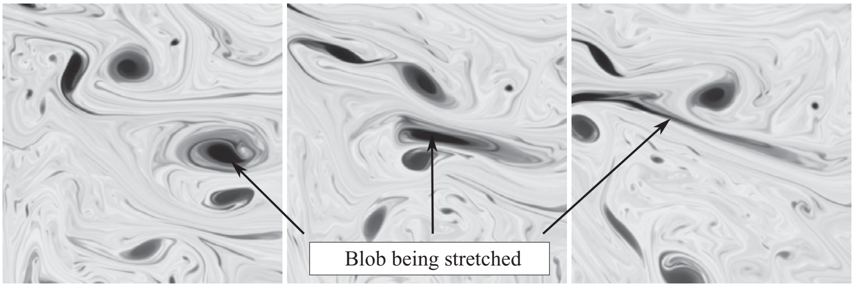

FIGURE 9. A large-scale vortex undergoing quasi-passive stretching.

get caught in a sufficiently strong strain they are stretched out in a quasi-passive manner; an example of this process is shown in figure 9. It seems likely that these passive large-scale structures contribute to the enstrophy flux at the large scales in the same way that the filaments contribute to the enstrophy flux at the small scales, and are therefore responsible for the $\partial E_\omega/\partial t \sim k^{-1}$ region at small wavenumbers.

5. Conclusions

We have demonstrated that in freely decaying two-dimensional turbulence the enstrophy spectrum has an extended k^{-1} region, and the enstrophy flux has a logarithmic dependence on wavenumber in the inertial range. This raises questions as to the validity of Batchelor's inertial-range hypothesis, which uses β as a surrogate for enstrophy flux, which in turn requires a uniform flux. Nevertheless, a simple reworking of his theory with β replaced by $\Pi_\omega(k_1)$ provides a good description of our results, and we interpret this as a manifestation of quasi-passive filaments being strained by coherent vortices.

We have also shown that the logarithmic enstrophy flux extends to wavenumbers $k_0 < k < k_1$, and that the enstrophy spectrum in this region has the form $kE_\omega = H(k) + A(t)$. We suggest that the dynamics for $k_0 < k < k_1$ are the same as when $E_\omega \sim k^{-1}$, but the steeper spectrum due to the coherent vortices masks this in $E_\omega(k)$.

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